A Design System for Orthogonal Pleat Tessellation

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Abstract

Origami is the traditional art of folding one sheet of paper into desired shape. With the development of origami mathematics theories and other techniques of origami design, the range of achievable shapes has increased considerably. Origami designers can design complex and highly detailed origami with one's own expertise. Origami researchers can also design those with origami design tools. On the contrary, inexperienced folders can hardly fold a sophisticated origami without attempts. Even with the origami tools, it is still too difficult for inexperienced folders to design origami.

Origami tessellations are made from a single piece of paper, which is folded in a repeating pattern. Pleat tessellation is a simple origami tessellation that contains only pleat folds. A pleat fold is formed by a pair of parallel mountain fold and valley fold. Despite its simplicity, pleat tessellation has its own advantages such as the easy generation of three-dimensional shapes.

In this paper, we study a very common form of pleat tessellations that contains only orthogonal pleats. For this particular kind of pattern, we want to simplify the universal origami theorems and to develop better algorithms for computing the folding sequences. We enumerate all its eight basic units which appear in crease patterns of orthogonal pleat tessellations and propose a new notation to rewrite the crease pattern. With this notation, users can easily input height information to design desired shapes. Although the definition of orthogonal pleat tessellation is very straightforward, it is not easy to fold it form the crease pattern. We propose a method to compute the folding sequence, normally not unique, from its crease pattern. In addition, based on this notation we also notice the existence of a combination of pleat units, which is referenced here as cyclic pleat pattern. We describe the characteristics of the cyclic pleat pattern and propose an algorithm to detect it in its crease pattern. Finally, we introduce a system that, as input, receives a matrix of our notations. It automatically detects inputted notations to prevent invalid notations. It, then, detect cyclic pleat pattern and output the graph with one possible folding sequences. This system can help people who are inexperienced in designing and folding origami to design three-dimensional origami. The notation proposed here also contributes with the visualization and further study of designing other three-dimensional origami.

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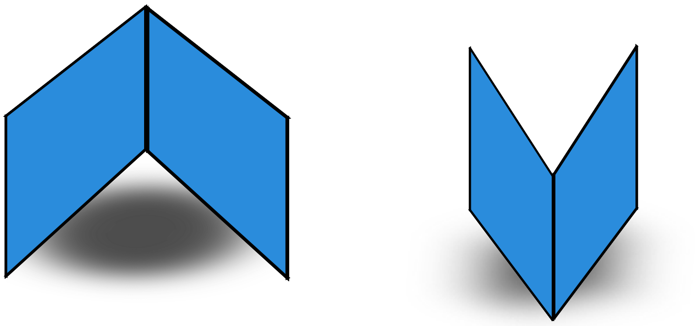
Chapter 1

Introduction

Origami is a traditional art of folding one piece of paper into desired shapes. Origami is originally a Japanese word which literately means paper folding. It is believed to appear along with the invention of paper. Although there is no evidence that shows origami is invented by Japanese, it developed and thrived in Japan and to be spread all over the world. Basically, it is made from a square of flat paper without any glue or cuts. Despite of its simplicity, innumerable geometric shapes have been produced by origami. Many of them can be very complex and abound in detailed features. This chapter describes some basic concepts and give a concise overview of modern origami. Many of the concepts will be used throughout this thesis.

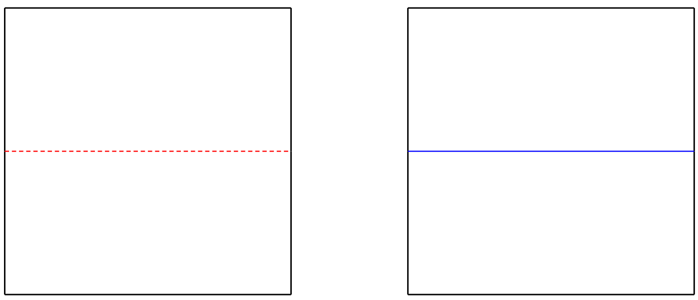
1.1 Basic origami concepts

There are many concepts in modern origami which cover from art to science. In this section we will discuss the most used terminology to build a base of this thesis and a specific pattern called orthogonal pleat tessellation which is the subject of this work. This section will not describe all concepts while other concepts will be introduced throughout the thesis. Because the most common action in origami is to fold, the basic terminologies of origami are simple and focus on folding. A *folding motion* of a piece of paper is a continuous motion of the paper from one configuration to another that does not cause the paper to stretch, tear, or self-penetrate. A snapshot of this motion at a particular time is called a *folded state*; in particular, we distinguish the *initial folded state* from which the folding begins and the *final folded state* at which the folding arrives. A *crease* is a line segment (or, in some cases, a curve) on a piece of paper. Creases may be folded in on of two ways: as a *mountain* fold, forming in a protruding ridge, or as a *valley* fold, forming an indented trough [DO07]. Figure 1.1 shows an example of mountain fold and valley fold. The type of a fold is relative, since one can switch a valley fold or a mountain fold to obtain the other type of fold. The usual convention is to display mountain folds as a dash-dot pattern, and valley folds as dashes only. In this thesis, we use red dashes for mountain folds, and solid blue lines for valley folds to avoid confusion (Figure 1.2).



(a) Mountain fold (b) Valley fold

Figure 1.1: Mountain fold and valley fold



(a) Mountain fold (b) Valley fold

Figure 1.2: Display of mountain fold and valley fold

1.1.1 Origami diagram and crease pattern

There are two common ways to document origami, *origami diagram* and *crease pattern*. Origami crease pattern has an ancient history, going back to the very beginning of origami itself. Origami crease pattern is the unfolded paper of origami. Origami was imparted and inherited by oral instructions or directly by showing the origami crease pattern. After mountain/valley notation was introduced, crease pattern with a more clear demonstration became available (Figure 1.3). Origami diagram was mainly designed by Akira Yoshizawa in 1950s and '60s and uses lines and arrows indicating the position of the folds and the movement of the paper [Rob04]. Figure 1.4 shows one origami diagram of origami crane. It shows detailed folding motions of each folded states. This method is present in majority of origami books because of its advantage of easy understanding. As a result, it greatly accelerate the spread of origami. However, the disadvantages of origami diagram is also evident. It is laborious to create a origami diagram. In addition, origami diagram is not suited for modern origami which often has more creases than traditional ones. Origami designed by mathematical method are usually complex with astonishing detail.

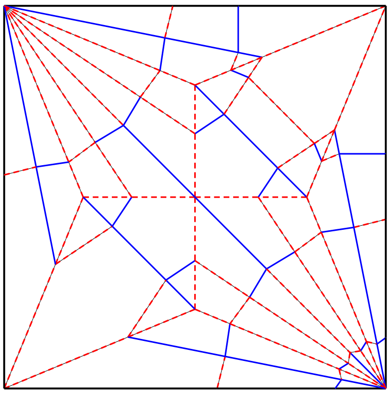


Figure 1.3: Crease pattern of origami crane

1.2 Motivation and goals

Despite the fact that origami has been studied for decades and many origami design tools have been developed, it is still not easy for inexperienced folders to not only design but also fold a sophisticated origami. We want to make more people design and fold their own origami without knowing complex origami theories and practicing repeatedly. However, a detailed origami usually contains many creases which is very difficult to fold. Tessellations as introduced in Section 2.4 are relatively easy to fold due to their repeated patterns. Among them there is a pattern called *Organic Tessellation* with back and forth mountain-valley folds can generate three-dimensional shapes (Figure 1.5). Consequently, we focus on this specific pattern and want to understand how this three-dimensional shapes generated so that we can use it purposely to assist origami design. In this essay, we reference *Organic Tessellation* as *Orthogonal Pleat Tessellation* because of its orthogonal pleat creases.

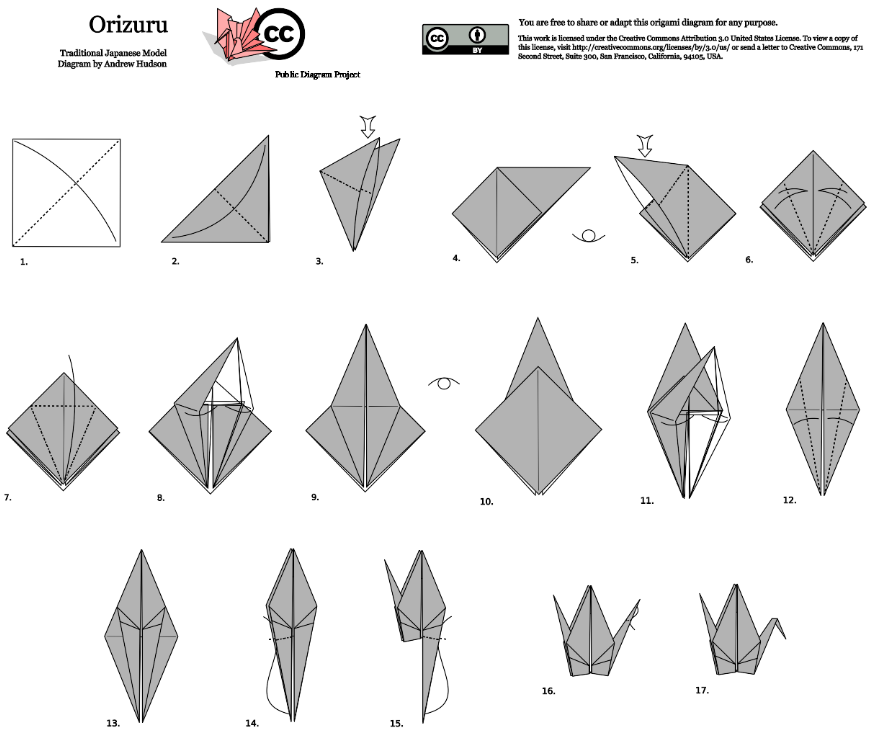


Figure 1.4: Origami diagram of origami crane

This work aims to produce a system that receives three-dimensional information directly as input and outputs folding sequences instead of a crease pattern. We expect that the proposed system help people who are not experienced in origami to design and fold three-dimensional origami.

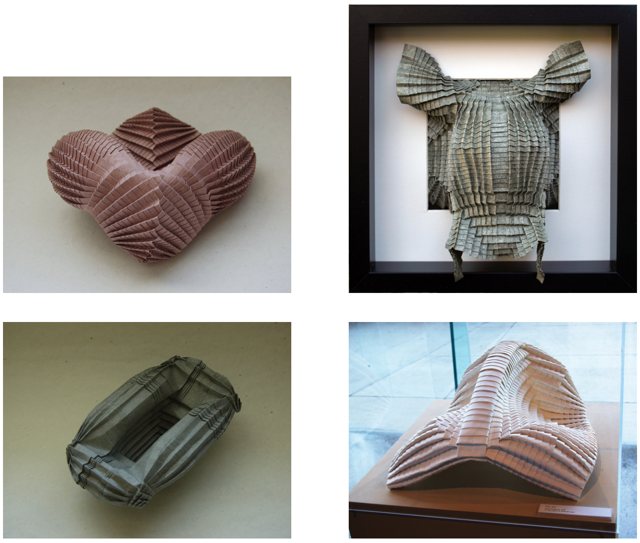


Figure 1.5: Orthogonal pleat origami
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1.3 Overview of proposed method and contributions

Because of the periodic characteristic of origami tessellation, we divide the crease pattern into basic unit to discover its unique property. We find there are only eight different basic units in orthogonal pleat tessellation. All of the eight basic units can be represented by two parameters. One parameter that we call as *height arrow* describe its height distortion while the other one that we call as *sequence arrow* describes its folding sequences. A system with design assistance and folding assistance is proposed based on the notations. There are five main contributions of this work.
(1) To uses distortions that caused by thickness and tension of paper purposely to design origami.
(2) A new notation of orthogonal pleat origami is proposed to simplify the crease pattern to a matrix. The rewritten matrix provides more comprehensible insight of the crease pattern and allows a $O(n)$ algorithm to compute the folding sequences.
(3) Some flat-foldable but not simply foldable patterns are found through proposed notation. It can be detected by the Depth-first search algorithm after converting the notation matrix to a directed graph.
(4) The proposed system takes notation as input and outputs graph with folding sequences. The input assistance of the system provide a real-time check to prevent user from inputting

invalid notation, which makes the origami to be flat-foldable.
(5) Some methods based on basic orthogonal pleat tessellation pattern are introduced to help design more complex origami.

1.4 Structure of this thesis

This chapter gives the overview of our work. Chapter 2 introduces more specific origami concepts and the related work of this study. Chapter 3 discusses properties of orthogonal pleat tessellation and introduces notation for basic units. Some examples are given to show by adapting the notation, one can predict the folded shape in a more clear way. Chapter 4 discusses the differences between general folding sequences and folding sequences of orthogonal pleat tessellation. Employing these differences, we show how a $O(n)$ algorithm can be implemented to compute the folding sequences. In addition, a flat-foldable but not simply foldable pattern that generated from combining basics units is introduced in Section 4.2. Chapter 5 shows the proposed system with its design assistance and fold assistance. Chapter 6 gives some instructions of four possible patterns in designing new orthogonal pleat tessellations. In this chapter we introduce a pattern that we call as *lock pattern* which can give great impact to the final shape of origami. We show its structure and point out the fact that proposed notation cannot predict folded shape correctly with this pattern.
Chapter 7 concludes this thesis, summarizing the results and limitations and discusses the possible directions for future research.

Chapter 2

Related Work

In this chapter, modern origami will be introduced, including origami mathematics, origami design, origami engineering and origami tessellation. To build a origami design system, we should know the computational origami method and the mathematical theory behind it. By reviewing existed origami design system, we are supposed to understand what they are superior of and what to improve. In the last section, one origami genre called origami tessellation will be introduced. Some features and existed tessellations will be discussed to show the interest part of this genre.

2.1 Origami mathematics

Modern origami has received a significant influence from mathematical researches. Origami mathematics also thrives as a subset of mathematics and gives profound insight of both origami art and origami science. This section will introduce two most clean and well-studied topics: Huzita-Hatori axioms and origami foldablity.

2.1.1 Huzita-Hatori axioms

Before origami mathematics has been studied, people had construct many geometric shapes using origamis. These constructions are obtained only by folding without any tools such as ruler. One can divide the side of a square in fractions such as thirds, fifths and ninths or trisecting an angle by folding manoeuvres [Hag02] [AL09]. *Huzita-Hatori axioms*, also knows as *origami axioms* is currently the most powerful known set of origami axioms.
Huzita [Huz89] formulated the first six axioms. Hatori [Hat02] found 7th axiom.
(1) Given two lines L1 and L2, to fold a line placing L1 onto L2.
(2) Given two points P1 and P2, to fold a line placing P1 onto P2.
(3) Given two points P1 and P2, to fold a line passing through both P1 and P2.
(4) Given one point P and one line L, to fold a line passing through P and perpendicular to L.
(5) Given two point P1 and P2 and one line L, to fold a line placing P1 onto L and passing through P2.

(6) Given two point P1 and P2 and two lines L1 and L2, to fold a line placing P1 onto L1 and placing P2 onto L2.
(7) Given one point P and two lines L1 and L2, to fold a line placing P onto L1 and perpendicular to L2.
Lang [Lan96] gave the proof of the completeness of the seven origami axioms. Lang has also created a computational tool to find approximations of arbitrary reference elements based on the origami axioms [Lan03]. The software that called *ReferenceFinder* can give origami manoeuvres to create the required reference elements.

2.1.2 Origami foldablity

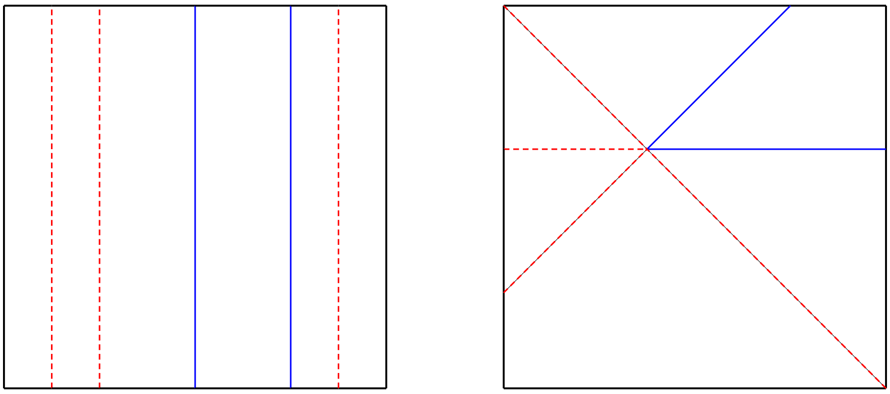
Origami foldablity generally describes whether crease patterns can be folded into origami that use exactly the given creases. Flat-foldability is the most well-studied foldability. Although we also list simply foldability as a sub-section, simply foldability is one kind of flat-foldability. However simply foldability directly relates to our work so that we will introduce it separately.

Flat-foldability

In order to discuss flat-foldability, we assume that paper to be used has zero thickness. A crease pattern is called *flat-foldable* if it can be folded into a flat shape finally. Here, we only focus on its final shape which means after every creases of a flat-foldable crease pattern have been folded with its mountain/valley assignment, the folded shape will be flat. The flat-foldablity problem can be divided into two sub-problem: *local flat-foldability* and *global flat-foldability*. Local flat-foldability concerns whether each vertex of given crease pattern can be folded flat locally or individually. While global flat-foldability is usually the same as flat-foldability of the crease pattern. The test for local flat-foldability have linear complexity while testing for global flat-foldability is NP-hard [BH96].
Two simple types of crease patterns appear exist in local flat-foldability problem (Figure 2.1). Figure 2.1a is parallel creases pattern (also called $1D$ pattern) which is proved to be solved in $O(n)$ worst-case time [DO07]. Figure 2.1b is single vertex pattern which related to two theorems, *Maekawa Theorem* and *Kawasaki Theorem*. Maekawa Theorem sates that in a flat-foldable single vertex mountain-valley pattern defined by angle $\theta_1 + \theta_2 + \cdots + \theta_n = 360°$, the number of mountains and the number of valley differ by ± 2 (Figure 2.2a). Kawasaki Theorem states that a single-vertex crease pattern defined by angles $\theta_1 + \theta_2 + \cdots + \theta_n = 360°$ if flat-foldable if and only if n is even and the sum of the odd angles θ_{2i+1} is equal to the sum of the even angles θ_{2i}, or equivalently, either sum us equal to $180°$ (Figure 2.2b). Another condition state that if an angle θ_i is a strict local minimum (i.e., $\theta_{i-1} > \theta_i < \theta_{i+1}$), then the two creases bounding angle θ_i have an apposite mountain-valley assignment in any flat-foldable mountain-valley pattern.

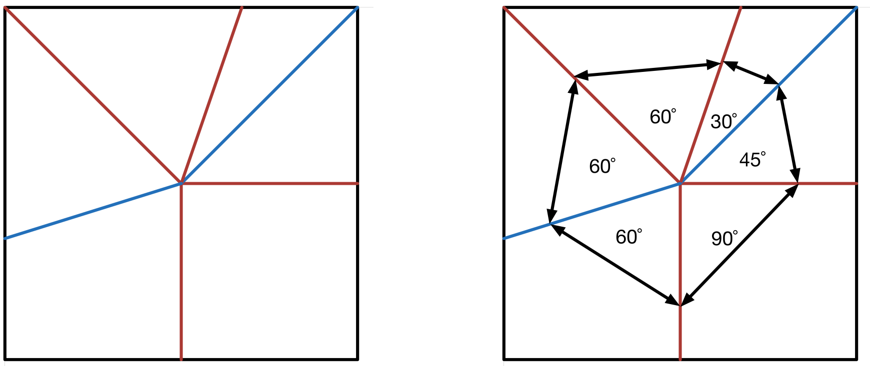
Simply foldability

A origami is *simply foldable* if and only if it can be folded with a sequence of *simple folds*. Demaine [DO07] give four restrictions for *simple folds*.



(a) Parallel (b) Single vertex

Figure 2.1: Two simple types of crease patterns
a: All crease are parallel and can be simplified to $1D$ pattern.
b: Creases form a single vertex. Maekawa Theorem and Kawasaki Theorem discuss about it.



(a) Maekawa Theorem (b) Kawasaki Theorem

Figure 2.2: Maekawa Theorem and Kawasaki Theorem

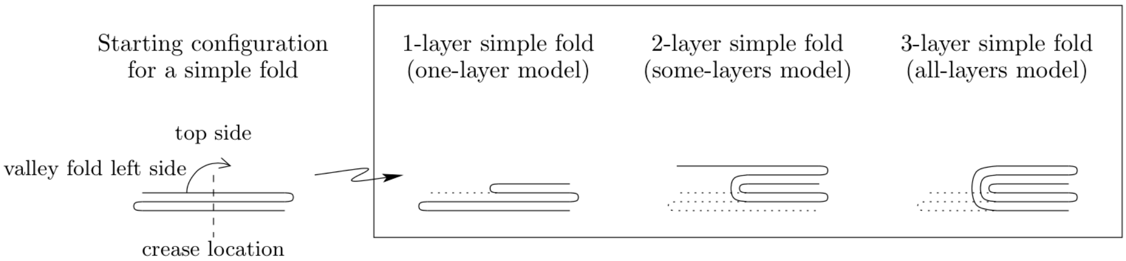
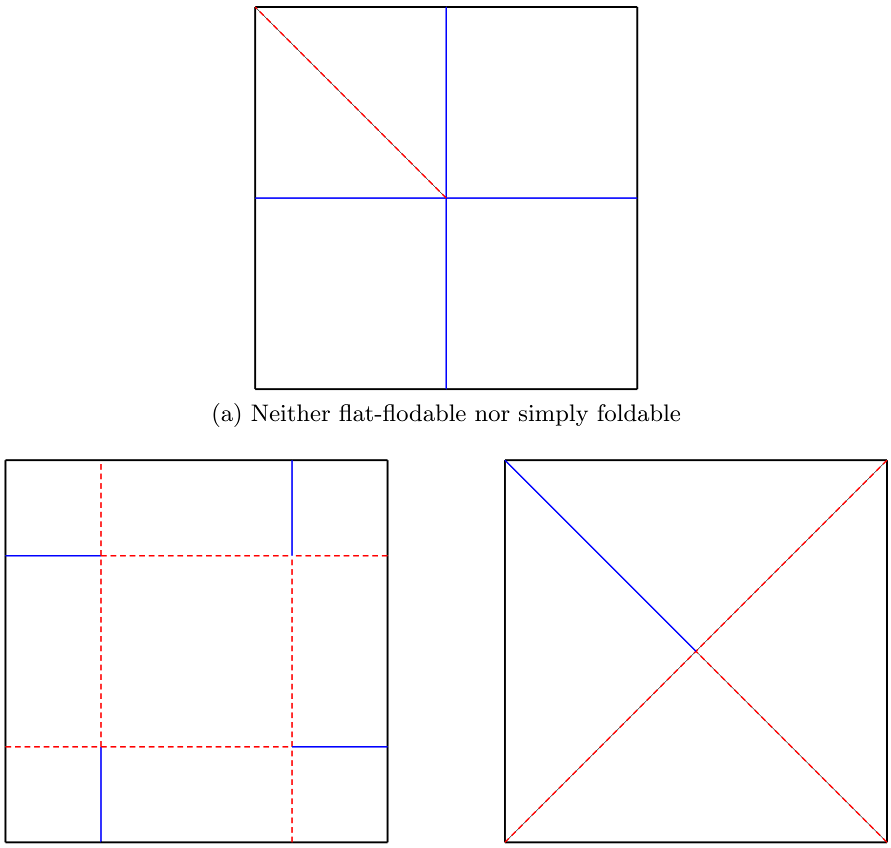


Figure 2.3: Three classes of simple folds [ABD+04b]

(1) Simple folds apply only to flat folded states, and map one flat folded state to another.
(2) Each simple fold is along a segment on the top of the folded state. This segment may or may not extend all the way across the silhouette of the paper.
(3) The fold is a rigid rotation of some layers of paper under the segment, avoiding self-intersection throughout 180° of rotation.
(4) The fold must respect the given crease pattern, folding only at creases, and according to the specified mountain-valley assignment.
Origami that contains only simple folds are called *pureland origami*. The first restriction shows that if one origami is simply foldable, it must be flat-foldable. Figure 2.4 shows three crease patterns which are similar to each other, but have different flat-foldability or simply foldability.
Three classes of simple folds may be distinguished, depending on how the layers are folded:
(1) *One-layer simple fold*: Just folds the top layer of paper.
(2) *All-layers simple fold*: Simultaneously folds all layers of paper under the crease segment.
(3) *Some-layers simple fold*: Folds some layers beneath the creasing segment, perhaps a different depth of layers along different portions of the crease.
It is established in Arkin [ABD+04b] that theses three models are all different (Figure 2.3).

2.2 Origami design

Origami design used to be creative work for origami artists. Using experiences and skills, origami artists have created numerous origami. With the development of origami mathematics, design origami with computer assistance became possible. In the past thirty years, many applications have been developed to help people design origami. Lang developed an algorithm called *tree method* and developed the design application *TreeMaker*. Bateman's application *Tess* aids the design of origami tessellations(See 2.4) [Bat02]. Tachi's application *Origamizer* generates a crease pattern that folds into a given polyhedron [Tac10]. Mitani's application *ORI-REVO* aids to design of 3D revolution shapes [Mit12]. Another application by Mitani is called *ORIPA* which simulates the folded shape of flat-foldable origami. However, most of previous research are based on ideal zero-thickness paper. In practical paper folding, properties of paper such as thickness and tension usually generate distortions to the final shape. Origami designed by these applications may fail to match



(a) Neither flat-flodable nor simply foldable

(b) Flat-foldable but not simply foldable

(c) Flat-foldable and simplpy foldable

Figure 2.4: Flat-foldability and Simply foldability

the desired shape due to these distortions. Conversely, these distortions also bring us some advantages. If distortions can be predicted, one can use this purposefully to design novel origami shapes. In this section, we will introduce ORIPA and TreeMaker.

ORIPA receives a crease pattern as input and outputs the flat shape of folded origami (Figure 2.5). User can also draw crease pattern in ORIPA's canvas as an input. ORIPA computers the folded form and also obtain the layer ordering. Because computing the overlap relation is a NP-complete problem, it only can be obtained by brute force. However, because of the small number of faces, it can be computed in a relatively short time, With the overlap relation, ORIPA renders the origami and shows the folded shape in a x-ray image.

TreeMaker is a design application that implements the *tree method* which is for the design of an origami base with arbitrary number, length and configuration of flaps (Figure 2.6). This method designs a *uniaxial base* and obtains the crease pattern in $O(NpolylogN)$ time where N is the number of creases(the output size) [DD01]. TreeMaker receives a skeleton of an insect as input and outputs the possible crease pattern.

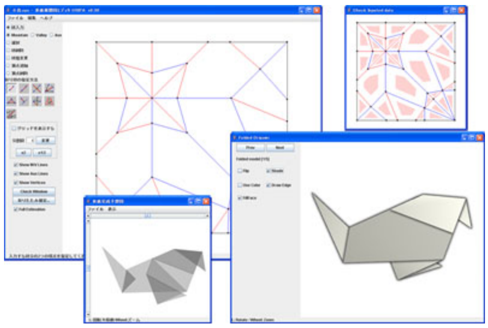


Figure 2.5: ORIPA[Mit12]

2.3 Origami engineering

Benefiting from the develop of origami mathematics, many other fields have been using origami techniques for its own purpose.

Miura fold has been considered as the pioneering application of origami engineering. Because Miura fold is a form of rigid origami which allows it to be used to fold surfaces made of rigid materials. The Japan Aerospace Exploration Agency, or JAXA used Miura fold to simulate large solar panel arrays. Ma [MY11] proposed a vehicle crash box using origami to enhance the crashworthiness of vehicles. It is to include energy absorption devices, designed to deform and absorb kinetic energy during a collision, at both the front and rear of the vehicles. Douglas use Origami DNA to create 3D shapes such as cubes and boxes. It was able to use Origami DNA techniques to create a clam-like cage which could carry and deliver drugs to specific target cells. The clam-like cage (nanorobots) had ”locks” which unzip when a target cell is found, thereby releasing drugs locally [DBC12].

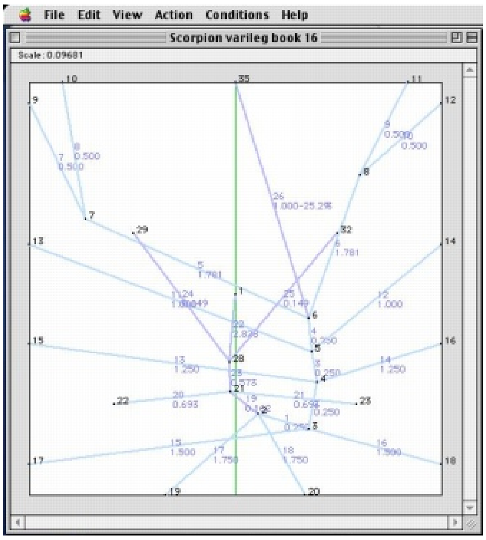
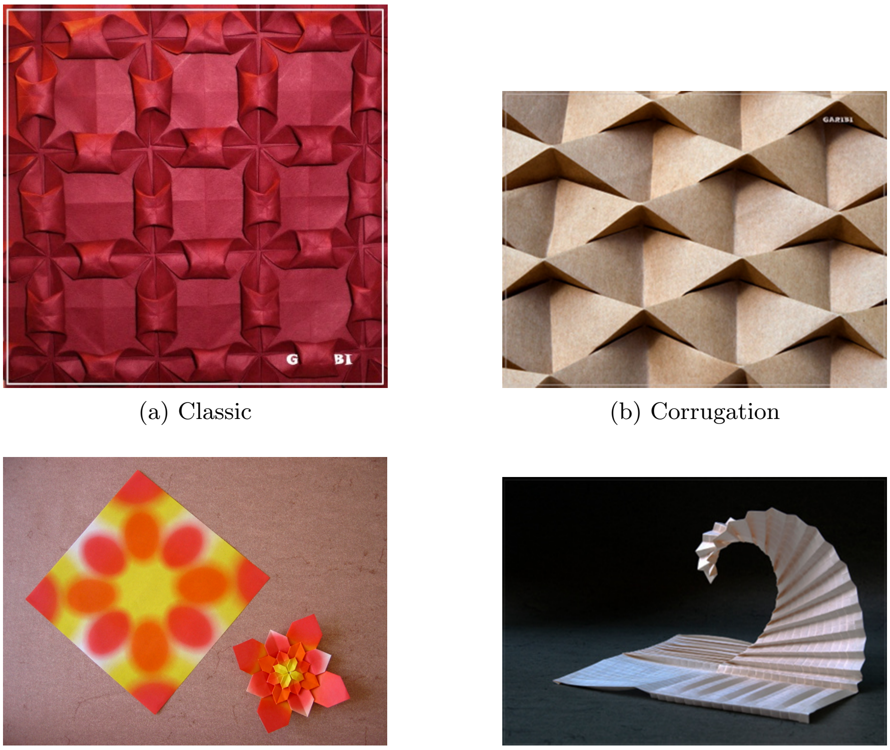


Figure 2.6: TreeMaker[Lan98]

2.4 Origami Tessellation

Roughly, an *origami tessellation* is a flat folding of a piece of paper based on a tessellation or tiling of the plane [GS87]. More generally, an origami tessellation might be defined to be any flat folding of the infinite plane in which no bounded region of the plane contains all of the vertices of the crease pattern [DD01]. Followed by Shuzo Fujimoto's Hydrangea tessellation in his book *Seizo Soru Origami Asobi no Shotai*, a wide range of origami tessellations have been designed. There is no standard method to categorize origami tessellations, Figure 2.7 lists four types of tessellations [Gar11]. Besides its repeated characteristic, Origami tessellations sometime use the transparent feature to create more detail other than just repeating.



(a) Classic (b) Corrugation

(c) Recursive (d) Organic tessellation

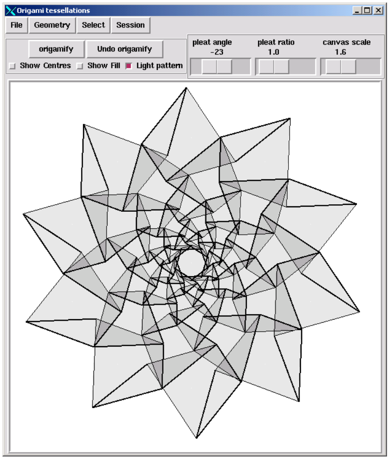
Figure 2.7: Four types of origami tessellations [Gar11]

2.4.1 Tess

Tess uses an algorithm that transforms a tiling of the plane into a flat-foldable crease pattern. To create a tessellation, one must scale and rotate each tile to make the whole shape flat-foldable. The algorithm takes two parameters, ratio of the lengths of the tile compared with the length of the edge of the baby tile (α/γ) and the pleat angle (ϕ),instead of taking the scale factor (α) and rotation angle of each tile (θ) [Bat02]. The relation between them is shown in Equation 2.1. n_1 and n_2 are the number of sides of the polygons.

$$\begin{aligned} \theta &= \arctan\left(\frac{1}{\tan\phi + \frac{\alpha/\gamma}{x\cos\phi}}\right); \\ x &= \frac{2\sin(\pi/n_1)\sin\pi/n_2}{\sin(\pi/n_1+\pi/n_2)}; \\ \alpha &= \frac{1}{\cos\theta + \sin\theta\tan(\theta+\phi)} \end{aligned} \tag{2.1}$$

Figure ?? shows the interface of Tess. One can use it to design classic tessellation.



(a)

Figure 2.8: Tess: origami tessellation software[Bat02]

2.4.2 Orthogonal pleat tessellation

Paul Jackson discovered *organic origami* in 1990's (Figure 2.9). The inspiration for organic origami comes from organic forms such as bacteria, seed heads and shells. Controversially for many origami purists, the paper is coloured with charcoal or dry pastel and sealed to create a surface with a matt lustre [Jac90]. Goran Konjevod extend organic origami to shapes other than organic, also even with other material such as cooper. Konjevod states that the experience of folding these pieces has helped him begin to understand how particular fold sequences interact and in a few cases he has been able to visualize the final shape before starting to fold [Kon06]. Therefore, no theory has been built to explain the three-dimensional shapes that generated by orthogonal pleat tessellation.

(a) (b)

Figure 2.9: Organic origami
©Paul Jackson [Jac90]

Chapter 3

Orthogonal Pleat Tessellation

In this chapter, we first introduce pleat tessellations, followed by orthogonal pleat tessellation, including its definition, basic unit and proposed notations. Orthogonal pleat tessellation is a subset of pleat tessellation. Although orthogonal pleat tessellation is simple, origami tessellations made by it are rather complex and usually generate three-dimensional shape.

3.1 Pleat Tessellation

3.1.1 Pleat Fold

Pleat fold is one of the most traditional and basic folds which also called Jyabaraori in Japanese. It is commonly used in clothing and upholstery to gather a wide piece of fabric to a narrower circumference [Pic57]. Two most common pleat folds are *knife pleat* and *box pleat*. Another typical application of pleat fold is folded fan (Figure 3.1*a*). The hinge part of a flexible drinking straw can also be considered as pleat folds(Figure 3.1*b*). We give a definition of pleat fold.

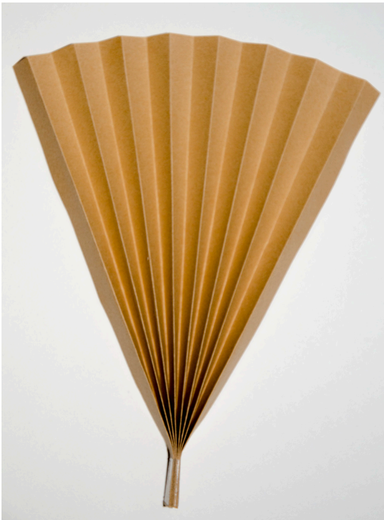
Definition (Pleat Fold)**.** *Pleat fold is a pair of parallel mountain and valley folds.*

The interval between mountain fold and valley fold is called *width* of the pleat fold. The angle between the pleat fold and the edge of the paper is called *angle* of the pleat fold. Therefore, we can use Equation 3.1 to present a pleat fold. Figure 3.2 shows crease patterns of two pleat folds.

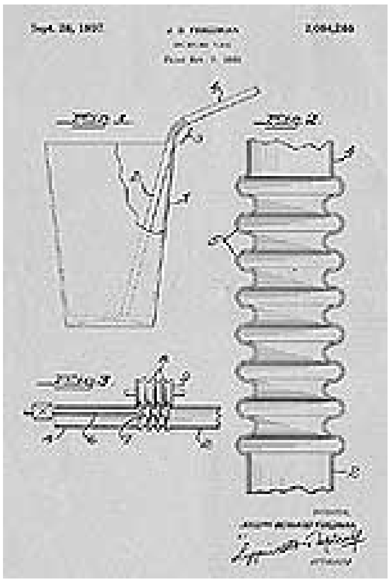
$$P_{w,a}^{\theta,p}(\theta \in [0,\pi], w \in \mathbb{R}, a \in \{M,V\}, p \in [0,1]) \tag{3.1}$$

θ denotes the direction. w denotes the width and the can be normalized by the size of paper. a denotes the arrangement of the pleat fold, namely whether mountain or valley fold is first clockwise. p denotes the position of interception between the pleat fold and the edge of paper. p is also normalized by the size of paper. Figure 3.2 shows two examples of pleat folds.

Knife pleat and box pleat are shown in Figure 3.3. The photos show the actual shape of

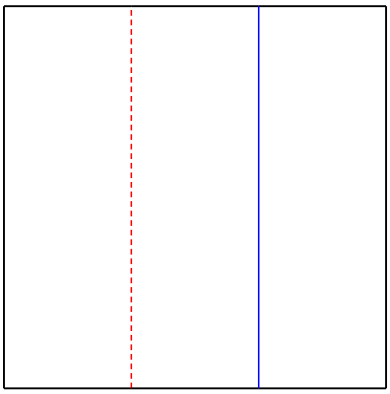


(a) Folded fan

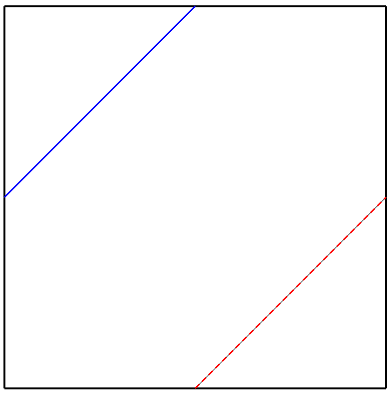


(b) Flexible drinking straw [BB02]

Figure 3.1: Folded fan and flexible drinking straw



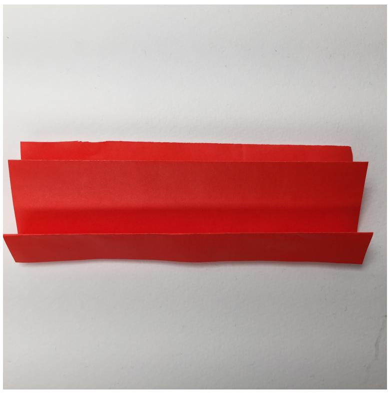
(a) $P^{\pi/2,}_{1/3,M}$



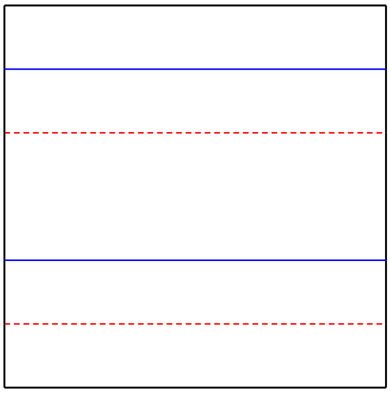
(b) $P^{\pi/4,}_{\sqrt{2}/2,V}$

Figure 3.2: Pleat folds

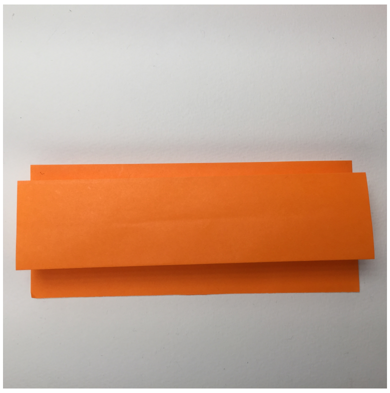
knife pleat and box pleat after practical folding.



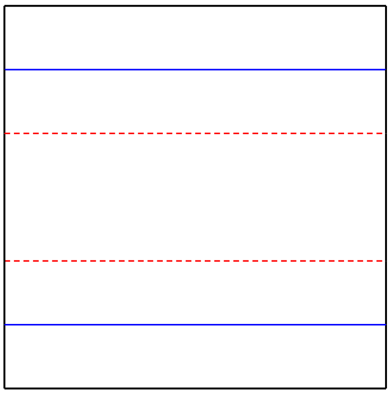
(a) Knife pleat



(b) $\{P^{0,1/4}_{1/6,M}, P^{0,3/4}_{1/6,M}\}$



(c) Box pleat



(d) $\{P^{0,1/4}_{1/6,M}, P^{0,3/4}_{1/6,V}\}$

Figure 3.3: Knife pleat and box pleat

If the direction of all pleat folds in a origami tessellation is same, the origami can be simplified as 1D origami. However, not all 1D origami can be restored back to origami tessellation of pleat folds with same direction. Section 2.1.2 introduces three types of simple folds. In pleat tessellations, every simple fold should be either one-layer simple fold or all-layer simple fold.

3.1.2 General Cases

By general cases, we mean that more than one direction and width exist in the crease pattern of a pleat tessellation. Equation 3.2 can be used to represent general cases of pleat tessellation.

$$T = \left\{ P^{\theta,p}_{w,a} : \theta \in [0, 2\pi], w \in \mathbb{R}, a \in \{M, V\} \right\} \tag{3.2}$$

Although executing a pleat fold requires two simple folds and each simple fold has only one direction, executing one simple fold doesn't ensure only adding one direction creases to crease pattern. Figure 3.4 shows the crease pattern after executing two pleat folds. The directions of creases are not two but three. With more pleat folded executed, the directions of creases will increase more quickly. In this essay, general cases will not be discussed.

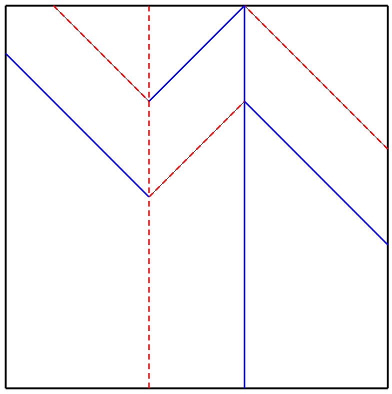


Figure 3.4: Crease pattern after executing two pleat folds

3.1.3 Map folding

Map folding problems seek to determine whether a given mountain-valley pattern on one piece of rectangle paper can be folded flat. Demine [DO07] gives the most basic version of map folding problem.

Definition (2D Map Folding Problem)**.** *Given a rectangle (the map) partitioned into an $n_1 \times n_2$ rectangle grid of squares, with each nonboundary grid edge assigned to be either a mountain or valley crease, can the map be folded flat into one square, respecting the creases.*

The computational complexity of map folding problem remains an open problem. Arkin [ABD+04b] consider a variation on the map folding problem in which the folding is restricted to simple folds. With this restriction, some $O(n)$ algorithms are used to determine the flat-foldability of one given crease pattern.
According to the definition of pleat fold, Arkin's variation on the map folding problem contains the foldability of orthogonal pleat tessellation. However, in Section 4.2 we discuss cases which appear due to our proposed notations. Those cases don't obey the simply foldability of pleat fold but show some interesting properties.

3.2 Orthogonal pleat tessellation

As discussed before when we add more directions of pleat into single origami, it will become complicated. Meanwhile, we notice that if we use two orthogonal directions of pleat, the origami will have both simplicity and the ability to form 3D shapes. We give a definition of orthogonal pleat origami to describe its properties.

Definition (Orthogonal Pleat tessellation)**.** *Orthogonal pleat tessellation is the tessellation formed by exclusive pleats folds that made in two orthogonal directions.*

The crease pattern of orthogonal pleat tessellation also only contains paired mountain and valley folds (Figure 3.5).

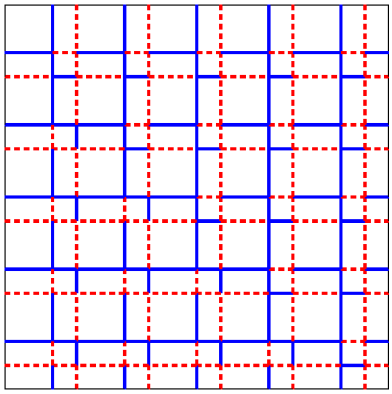


Figure 3.5: Crease pattern of orthogonal pleat tessellation

3.2.1 Basic unit

Although the crease pattern is much more clear than those of general cases, it still not make sense to most of users. By the recurrent characteristic of tessellation, we find that crease patterns of orthogonal pleat tessellations can be divided into several $3w \times 3w$ units. We consider those as basic units of orthogonal pleat tessellation.

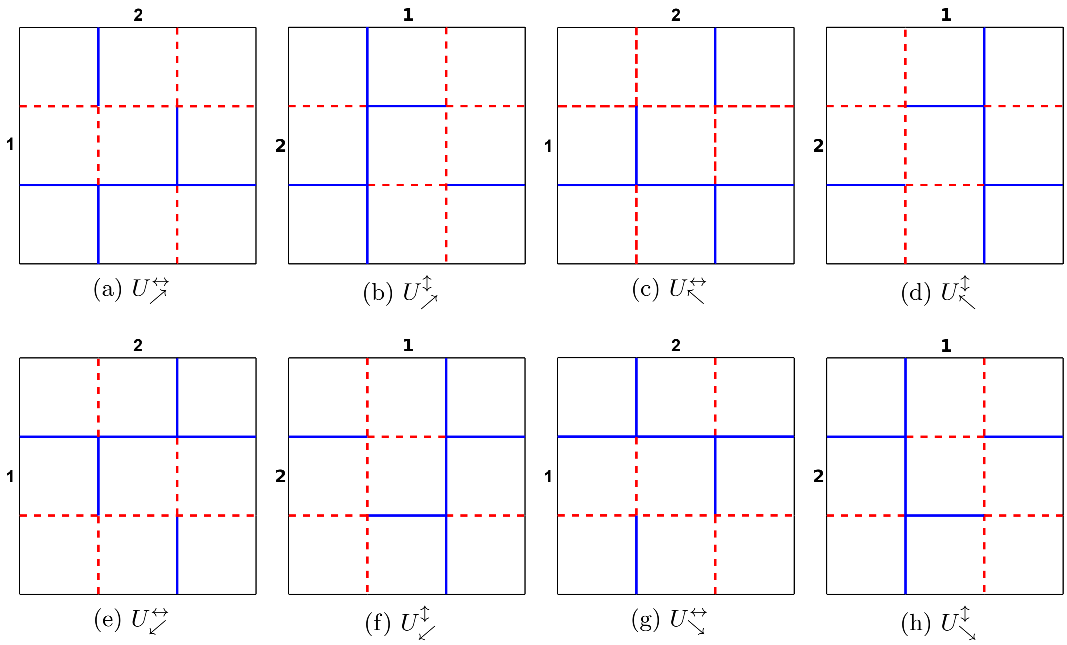
3.2.2 Notation

Without loss of generality, we can fix the vertical and horizontal direction of pleat to form a basic orthogonal pleat tessellation unit. We found that in orthogonal pleat origami of two pleat folds with fixed width, only 8 different basic units exist. There are two ways in which one can assign mountain and valley for a single pleat: with a mountain fold on the left or on the right. For a pleat tessellation unit, there are $2 \times 2 = 4$ types of combination. We can either fold vertical or horizontal direction at first. Therefore, we can only get $4 \times 2 = 8$ different basic units.

We introduce a notation for pleat units.

$$U^{d}_{\theta,g}(\theta \in \{\nearrow, \nwarrow, \searrow, \swarrow\}; d \in \{\leftrightarrow, \updownarrow\}, g = w_h/w_v)$$

θ denotes the direction of generated height in the unit. Hence, θ always have for different values. g denotes the gradient of the basic unit. It is defined as the quotient between widths of vertical and horizontal pleat folds. With our constraints, g is 1. One can make g to any value by modify the ratio of widths of two orthogonal pleat folds. d denotes the folding sequence of the unit. Namely, whether x-axis or y-axis has been folded first. Therefore, we call θ the height arrow and d the sequence arrow. We enumerate all 8 basic units that appear in orthogonal pleat origami using our notations (Figure 3.6). Each unit has two vertical folds and two horizontal folds. Theoretically, each unit is folded into a flat square. However, practically the folded one will form an open angle at certain direction. This open angle makes the origami generate height. Figure 3.7a, b shows the height direction after folding one pleat. The reddish the lower. Figure 3.7c shows the height direction of a basic orthogonal pleat origami unit. In this example, the height direction is from top-left to bottom-right.



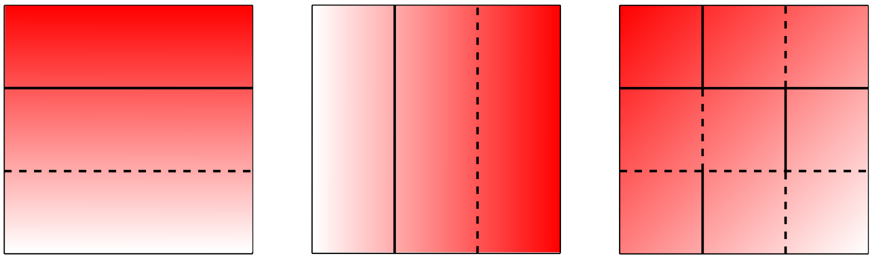
(a) $U^{\leftrightarrow}_{\nearrow}$ (b) $U^{\updownarrow}_{\nearrow}$ (c) $U^{\leftrightarrow}_{\nwarrow}$ (d) $U^{\updownarrow}_{\nwarrow}$

(e) $U^{\leftrightarrow}_{\swarrow}$ (f) $U^{\updownarrow}_{\swarrow}$ (g) $U^{\leftrightarrow}_{\searrow}$ (h) $U^{\updownarrow}_{\searrow}$

Figure 3.6: Eight basic pleat units.
Dash line: valley fold; solid line: mountain fold.
Number: folding sequences.

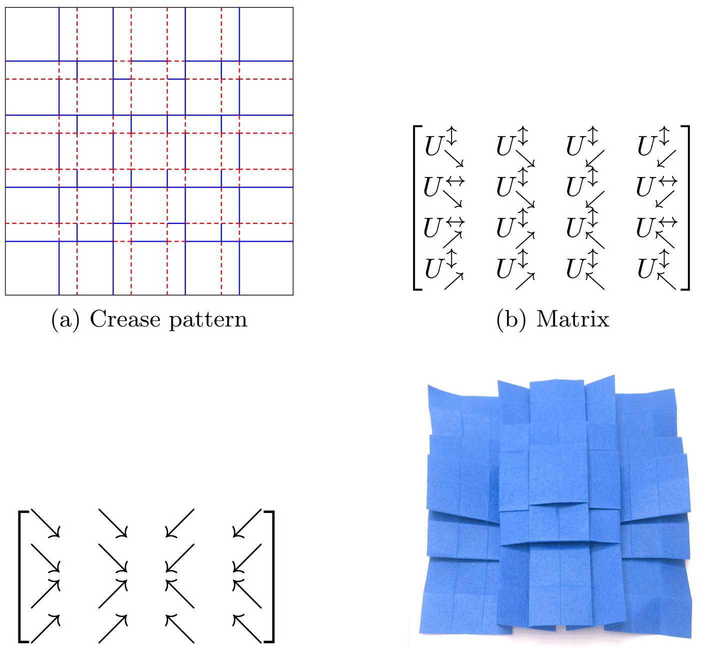
Pyramid is one of the very common pleats origami patterns (Figure 3.8a) . With our notation, this pattern can be represented by the matrix shown in Figure 3.8b. The height arrows give a good estimate of the folded shape.

On the other hand, the user can use our notation to assign 3D information which is more intuitive than a crease pattern.



(a) (b) (c)

Figure 3.7: Pleat folds prevent layers from being folded into flat sheet.
Solid line: valley fold; dash line: mountain fold.
Height color: the reddish the lower.



(a) Crease pattern
(b) Matrix
(c) Arrows
(d) Photo of the folded shape

Figure 3.8: Pyramid pattern

3.2.3 Combination of basic units

In order to design pleats tessellation, we want to know the restriction in combining these basic units. Obviously, mountain and valley fold cannot join on the same direction. Considering the flat foldability, we have the following restrictions.

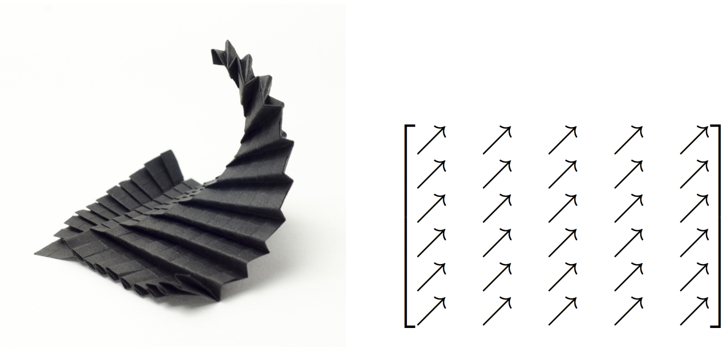
With $U_{\theta_i}^{d_m}$ and $U_{\theta_j}^{d_n}$, let $\Delta\theta = |\theta_i - \theta_j|$:
(1)$\Delta\theta = 0$: can connect at both x and y axes;
(2)$\Delta\theta = \pi/2$: can connect only at x axis;

(3)$\Delta\theta = \pi$: cannot connect at neither x axis nor y axis;
(4)$\Delta\theta = 3\pi/2$: can connect only at y axis;

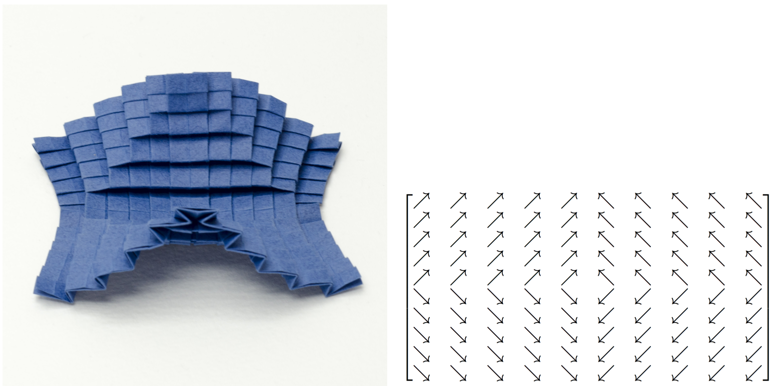
As we can see, basic units cannot be combined arbitrarily. From the geometrical perspective, the restriction (3) shows that sudden changes in the gradient are not allowed, as we cannot have adjacent units with opposite direction of height arrow.

3.3 Examples

We give some examples with our notation (Figure 3.9). Figure 3.9*a* is 1/4 of a pyramid. Figure 3.9*b* rearranges four parts of a pyramid to form a new shape.



(a)



(b)

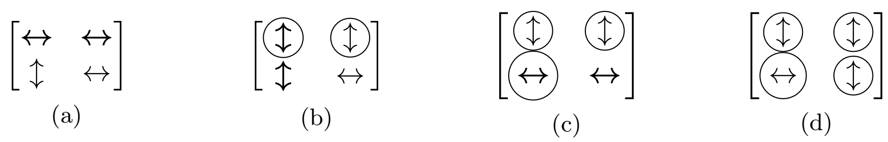
Figure 3.9: Examples of designs and extractions of height arrows

Chapter 4

Folding Sequences

4.1 Common pattern

There are two common ways to document origami. One is crease pattern and the other one is origami diagram. Folding an origami from its crease pattern requires experience and skills. Akitaya proposed a method to generate origami diagrams from a crease pattern [AMKF13]. For orthogonal pleat origami, we propose a more simple method. Because each basic unit is folded by two pleats, on x-axis and y-axis, and the sequence arrow denotes whether x-axis or y-axis has been folded first. After one combining basic units to design new tessellation, it is easy to know that the row or column with same sequence arrows can be folded by one simple fold. After finding the foldable row or column, we switch the corresponding sequence arrows to create a new matrix. The new matrix represent the current condition after the fold. Repeat the action until all the sequence arrows are switched. Figure 4.1 shows how a 2 by 2 matrix adapts the algorithm. **Notice that each arrow only need two step to be folded, therefore once an arrow has been switched, it cannot be switched again.** We use cycles to show which arrows have been switched.
First in Figure 4.1a, arrows at $(1, 1)$ and $(1, 2)$ are all $\leftrightarrow$. Therefore, we change the direction of these arrows to $\updownarrow$ (Figure 4.1b). Then arrows at $(1, 1)$ and $(2, 1)$ are all $\updownarrow$, we change the direction to $\leftrightarrow$ (in bold). Because arrow at $(1, 1)$ has been changed at first step, we only change arrow at $(2, 1)$ (Figure 4.1c). Similarly, $(2, 1)$ and $(2, 2)$ with the same $\leftrightarrow$, only $(2, 2)$ is changed (Figure 4.1d). Because Figure 4.1d is exactly the opposite of Figure 4.1a, the procedure is done.



(a) (b) (c) (d)

Figure 4.1: Switch arrows to obtain folding sequence arrows in bold will be switch in the next step.
Arrows with circle have been switched once.

Pseudo code is shown in Algorithm 1.

Algorithm 1 Update Folding Sequence

Input: Matrix of Sequence Arrows
Output: Folding Sequence

```
function GenerateFoldingSequence(matrix)
    foldablity ← True                                   ▷ Suppose origami is foldable.
    for unit ∈ matrix[ : i] do
        foldablity ← foldablity & ∼ unit.openDirection
        if (∼ foldablity) then
            break
        end if
    end for
    for unit ∈ matrix[i : ] do
        foldablity ← foldablity & unit.openDirection
        if (∼ foldablity) then
            break
        end if
    end for
    if foldablity then
        unit ← mark                        ▷ Mark the unit that has been switched.
        foldingSequence+ = newFold         ▷ Add one fold to folding sequence.
    end if
return foldingSequence
end function
```

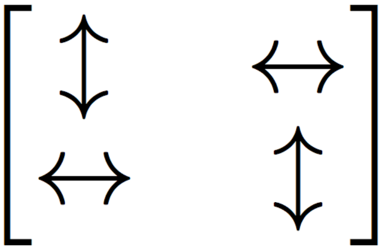
4.2 Non-simply foldable pattern

4.2.1 Cyclic pleat pattern

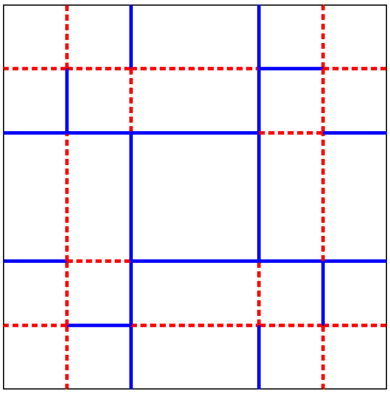
If there is no row or column with the same sequence arrows, the origami cannot be folded by simple folds. Edmonds observed that orthogonal 2D mountain-valley patterns may be flat-foldable but not by simple folds and Arkin gave some examples of such case [ABD$^+$04]. We also find an example that match the case which we call as cyclic pleat pattern. The simplest cyclic pleat pattern is formed by four basic units whose open direction arrows are placed as shown in Figure 4.2*a*. Its crease pattern is shown in Figure 4.2*b*. We define the cyclic pleat pattern as below.

Definition (cyclic pleat pattern)**.** *Cyclic pleat pattern is a pattern of pleat folds that has a cycle in the layering order.*

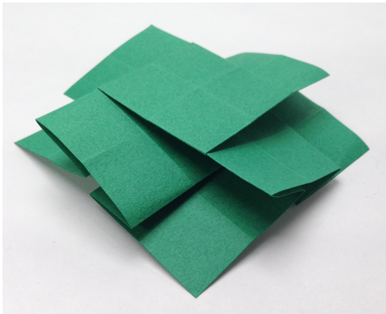
The cyclic pleat pattern will make the whole origami lose its simply foldability. Although the origami remains foldable, not knowing the position of cyclic pleat pattern can make the



(a) Sequence arrows



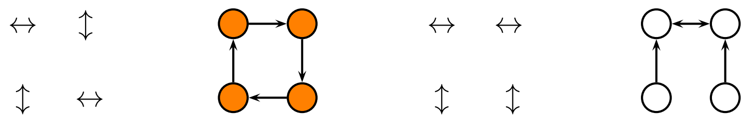
(b) Crease pattern



(c) Photo

Figure 4.2: cyclic pleat pattern

fold procedure much more difficult.
The matrix of sequence arrows can be converted into a connected graph: each arrow is a node in the graph and $\updownarrow$ connects its upper and lower node, $\leftrightarrow$ connects its left and right node. Figure 4.3a is a cyclic pleat pattern, Figure 4.3b forms a connected cycle. Figure 4.3c is not a cyclic pleat pattern, hence there is no cycle in its graph (Figure 4.3d).

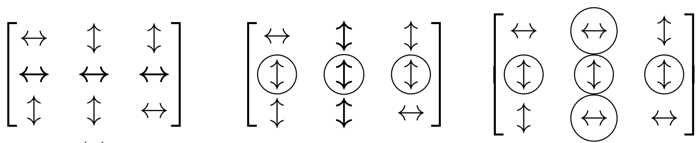


(a) (b) (c) (d)

Figure 4.3: Two examples of converted graph

Conjecture 1. *A crease pattern contains a cyclic pleat pattern if at least one of its folding states generates a graph that contains a directed cycle.*

Notice that since a cyclic pleat pattern is determined only by sequence arrows, changing height arrows would not break a cyclic pleat pattern. Therefore, we can use a Depth-First-Search algorithm to find a cycle in corresponding graph.



(a) (b) (c)

Figure 4.4: Not so obvious cyclic pleat pattern

In addition, this algorithm also holds when arrow switches after several folds (Figure 4.4). Before we switch the arrows, there is no cycle of sequence arrows (Figure 4.4a) . After two switches, making two simple folds, one cycle forms (Figure 4.4c). The reason why the not obvious cyclic pleat pattern exist is that crease pattern only records the creases after the entire folding procure is done. While the simply foldability only describe whether the entire origami is simply foldable, not telling how much one origami can be folded by simple folds. Maybe the origami lost the simply foldability by the very first step or the very last step. By knowing the exactly which step will fail to remain simply foldable, we can deal with these problems more efficiently.
Figure 4.5 gives some examples of a cyclic pleat pattern. Figure 4.5a, b are common cases that cyclic pleat pattern exists. Figure 4.5c, d shows that it is possible to have more than one cyclic pleat pattern at the same folding state.

$$\begin{bmatrix} \leftrightarrow & \leftrightarrow & \updownarrow \\ \updownarrow & \ast & \updownarrow \\ \updownarrow & \leftrightarrow & \leftrightarrow \end{bmatrix}$$

(a)

$$\begin{bmatrix} \ast & \ast & \ast \\ \ast & \leftrightarrow & \updownarrow \\ \ast & \updownarrow & \leftrightarrow \end{bmatrix}$$

(b)

$$\begin{bmatrix} \leftrightarrow & \updownarrow & \leftrightarrow \\ \updownarrow & \updownarrow & \updownarrow \\ \updownarrow & \leftrightarrow & \updownarrow \end{bmatrix}$$

(c)

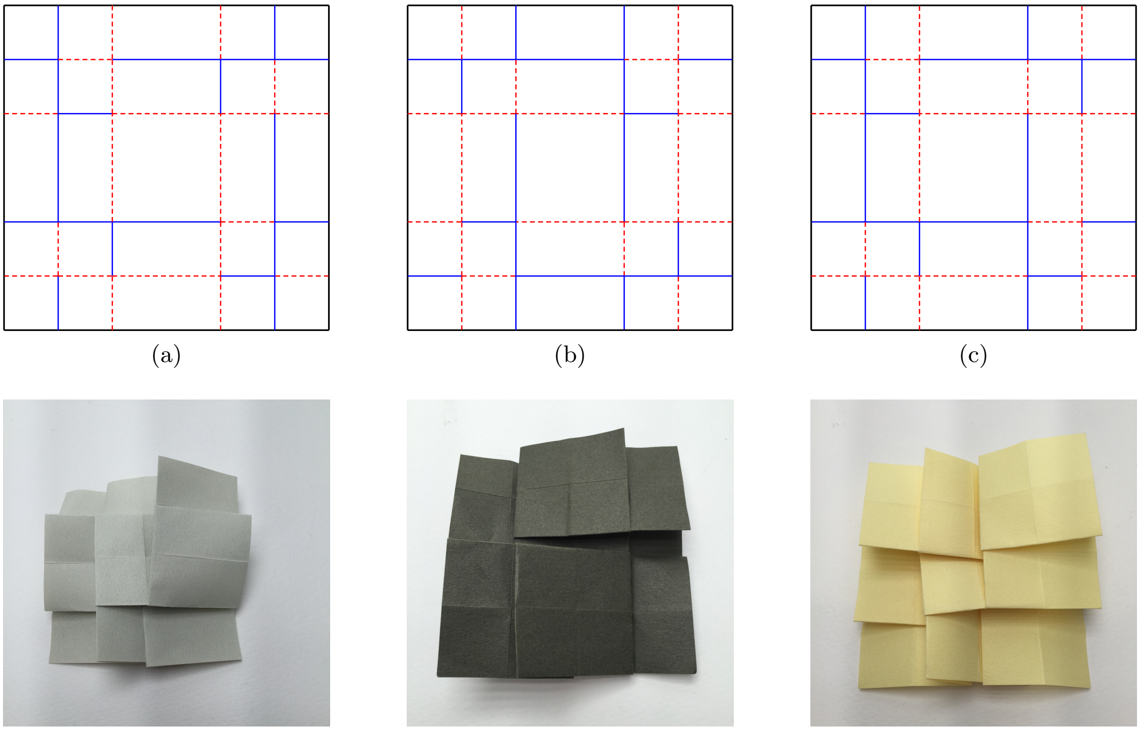
$$\begin{bmatrix} \leftrightarrow & \leftrightarrow & \leftrightarrow & \updownarrow \\ \updownarrow & \leftrightarrow & \updownarrow & \updownarrow \\ \updownarrow & \updownarrow & \leftrightarrow & \updownarrow \\ \updownarrow & \leftrightarrow & \leftrightarrow & \leftrightarrow \end{bmatrix}$$

(d)

Figure 4.5: Examples of cyclic pleat patterns

4.2.2 Variations of cyclic pleat pattern

If we change height arrows of cyclic pleat pattern, we can obtain some other interesting patterns. These patterns do not have cyclic layer relations, but are still not simply foldable. Figure 4.6 shows crease patterns of three possible variations of cyclic pleat pattern.



(a) (b) (c)

(d) (e) (f)

Figure 4.6: Variation of cyclic pleat patterns

Chapter 5

Implementation and Results

In this Chapter, the structure of proposed system will be introduced, including how we build the user interface for inexperienced folders. In the user interface section, we will discuss how we try to build a user-friendly interface to let inexperienced users to create their own origami tessellation.

5.1 User interface

Since the system is designed for inexperienced folders, it should be easily comprehensible and manipulable. Similar to most other design systems as introduced in Section 2.2, this system adopts the method to be divided into two parts, the design assistance and folding assistance. Design assistance aims to help to convert one's design idea to computational data which also to be the input of the system. Folding assistance aims to help to convert the computed result to executable origami instructions which also to be the output of the system.

5.1.1 Design assistance

As the input interface of the system, design assistance is expected to be straightforward with a relative gradual learning curve. Heightmaps are commonly used to store three-dimension data for display in computer graphics. However, less height details are in pleat orthogonal tessellation. Proposed height arrows can represent three-dimensional information of basic units. Hence, we develop two partitions in design assistance, the input panel and the paper canvas (Figure 5.1). Paper canvas contains the space to be filled in with basic units in input panel.
To develop a user-friendly interface, we want to eliminate user's unnecessary and inconvenient actions. Combinations of random basic units could make origami to be not flat-foldable. A real-time check function is developed to deal with this problem. A slide bar is also used to help user set up the size of canvas.

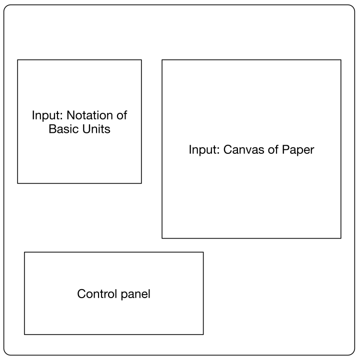


Figure 5.1: Structure of design assistance

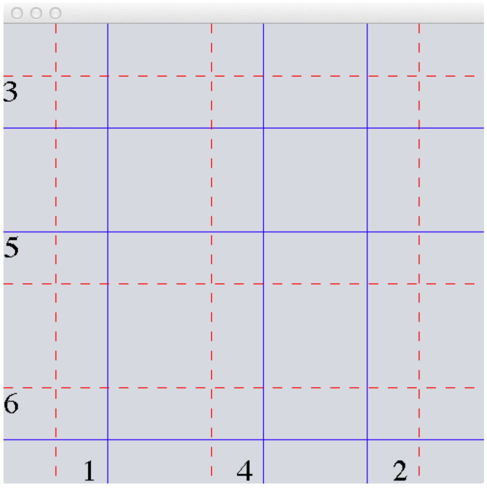
5.1.2 Folding assistance

The folding assistance is developed to output the instruction of how to fold the designed origami. Because of the simply foldablity of orthogonal pleat tessellation, we abandon the common crease pattern method and origami diagram method. Instead, folding sequences is developed to be directly shown as numbers that represent the folding order along with the crease pattern. However, with more creases exist which is very common in large scale tessellations, the crisscross mountain-valley creases become difficult to distinguish. To solve this problem, we uses the exactly mountain or valley fold that user is supposed to execute instead of the actually creases in crease pattern.

5.2 Design System

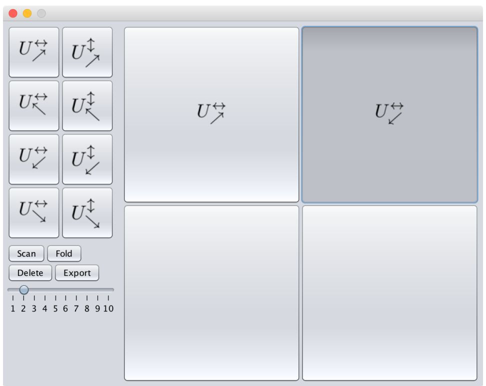
Figure 5.2*a* shows the interface of our application. The user are only allowed to input valid notation based on the combination rules. During the input procedure, this function detects possible invalid inputs and prevent them immediately (Figure 5.3). After inputting notations, the application can output the folding sequences (Figure 5.2*b*). The output for folding sequences is not crease patterns but only show which fold should be performed at certain position.

(a) Input



(b) Output

Figure 5.2: Interface of the application



(a) Invalid inputs

(b) To be prevented

Figure 5.3: Real-time check function

Chapter 6

Instructions on Designing Orthogonal Pleat Tessellation

Although orthogonal pleat tessellation can generate three-dimensional shape, the shapes are still monotonous compared to other detailed origami artworks. We explore to find some patterns which can be used to design origami. In each section of this chapter, we describe the pattern and show some examples to demonstrate the design methods. These patterns are believed to be combined to create more complex shapes.

6.1 Pyramid pattern

Pyramid pattern is the pattern with single peak. The center of pyramid pattern is highest. The smallest pyramid pattern needs four basic units and the arrangement is shown in Figure 6.1. All basic units in Figure 6.1 have the same gradient. However, basic units with different gradients also can be combined to form a pyramid pattern. Figure 6.2 shows an example. The gradients are all different.

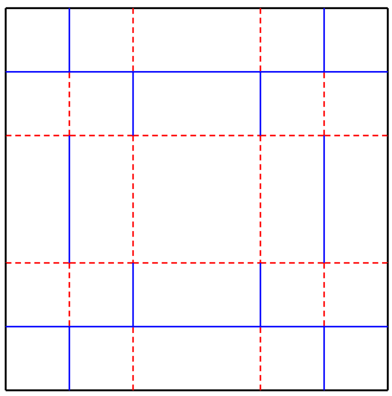


Figure 6.1: Smallest pyramid pattern

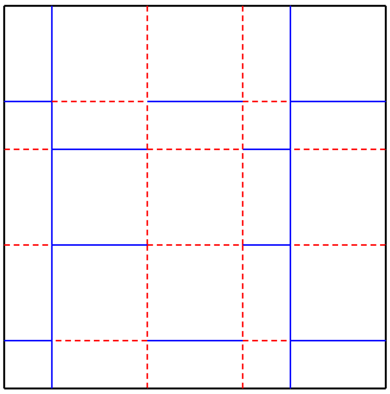


Figure 6.2: Pyramid pattern contains basic units with different gradient

6.1.1 Arc pattern

Arc pattern is the pattern that generate arc-like shape. The crease pattern of this pattern is one quarter of that of the pyramid pattern. Theoretically, the smallest arc pattern is one basic unit, and more basic units with same θ are added, the final shape should be a arc not a slope. The reason that forming a arc not a slope is that the edge parts of arc pattern diverge. While the corresponding part of pyramid pattern converge due to the tense of paper. Figure 6.3 shows one example with arc pattern.

Figure 6.3: Tessellation with arc pattern

6.2 Lock pattern

Lock pattern is the pattern that locks the paper to make them closer. The notation of a lock pattern is very simple: a $\leftrightarrow$ next to a $\updownarrow$. Lock patterns are so common in orthogonal pleat tessellation that may not be considered as a special pattern. However using lock patterns in on edge can greatly change the shape of orthogonal pleat tessellation (Figure 6.4). In this example, an arc pattern transforms to a shell-like shape due to lock pattern on edges.

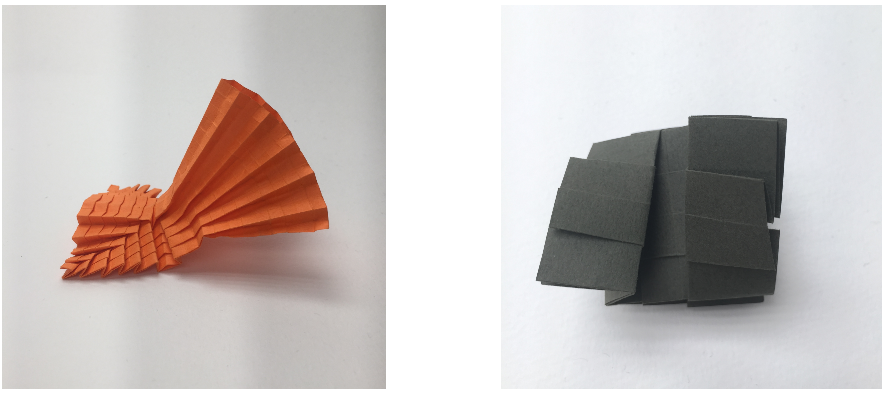
Even we look at the notation matrix of these two tessellations, all height arrows are the same except those on the edge.

(a) Arc pattern (b) Arc pattern with lock pattern

Figure 6.4: Lock pattern

6.3 Non-simply foldable pattern

Non-simply foldable pattern is introduced in Section 4.2. The main property of this pattern is the difficulties of folding and unfolding. This property can be used to make tessellation more stable at where the non-simply foldable pattern is (Figure 6.5*a*). Since the non-simply foldable pattern can create door-like shape, another design method is that to use it cover other parts of tessellation. Figure 6.5*b* shows one example that covers a small pyramid pattern inside the non-simply foldable pattern.



(a) Make more stable (b) Cover other patterns

Figure 6.5: Non-simply foldable pattern

Chapter 7

Conclusion and Future Work

7.1 Conclusion

In this paper, we study orthogonal pleat tessellation under the consideration of paper thickness. Our work uses distortions that caused by thickness and tension of paper purposely to design three-dimensional origami. With the repeatable feature of tessellation, eight basic units of orthogonal pleat tessellation are found. A new notation of basic units is proposed. Using the proposed notation, the crease pattern can be converted into a matrix which provides more comprehensible insight of the crease pattern and allows a $O(n)$ algorithm to compute the folding sequences. Some flat-foldable but not simply foldable patterns are found by arranging proposed notation. It can be detected by the Depth-first search algorithm after converting the notation matrix to a directed graph. A design system is proposed which takes notation as input and outputs graph that contains folding sequences with folding order numbers. The input assistance of the system provide a real-time check to prevent user from inputting invalid notation, which makes the origami to be flat-foldable. At last, some instructions based on basic orthogonal pleat tessellation pattern are introduced to help design more complex tessellations.

7.2 Limitation and Future work

Limitation: There are three major limitations of our work.
(1) Although this study considers the physical properties of paper, there are no precise physical models applied to provide the absolutely correct predictions. As a result, proposed notations can be applied to many patterns to predict relatively similar folded shapes, tessellations that do not suit our method still exist. For example, tessellations that contains lock patterns usually forms the very different shapes compared to predicted ones. Another example, distortions are not only generated in z-axis (generating height), xy-plane which should be a square is torn to be a rhombus.
(2) All the tessellations shown in this study are made by paper of fixed thickness [Toy14]. And all these tessellations are folded by human. It does not ensure that every creases fit the exactly positions where those should be in the crease pattern. This situation becomes more

noteworthy while using more thicker paper. The imprecise folding procedure is considered to magnify distortions in final folded shape. However, this study did not take this factor into account.
(3) By converting the notation matrix to directed graph, cyclic pleat patterns can be detected. However, the system will not be able to give the complete folding sequences. (The partial folding sequences can be outputted.) In fact, the complete folding sequences of non-simply foldable tessellations can not be represented only by folding orders. This brings folding difficulties to inexperienced folders. Even for experienced folders, it still require to bend the paper to complete the folding. The proposed system did not provide possible solutions for this situation to make the folding procedure more easily.
Future work Three possible future work will be discussed as following.
(1) The proposed notations are considered to provide good estimation of the final shape. In order to obtain more correct shape, more factors need to be considered. Not only to improve the correctness but also to adapt the model to more general orthogonal pleat tessellations, such as tessellation with lock patterns. In the meantime, a three-dimensional simulation may also be added to the system to provide a real-time estimation.
(2) Transparency and duo color (usually the front-side color and the back-side color of paper) are used in tessellation design. Although in orthogonal pleat tessellation, three-dimensional shapes are focused, details such as duo color can be added to increase the variation. Non-simply foldable patterns bring difficulties to folding, but also bring the variation on design. A new approach that shows the folding instructions of non-simply foldable patterns may be added in order to improve out proposed system. The new approach must be adopted, possibly modeling the bending of the paper, in order to produce results that resemble the actual paper folding.
(3) Combing different sizes of basic units has not been studied in this essay. A more strict combination rule should be found. On the other hand, as long as the new patterns are easily to be folded for inexperienced folders, some patterns which may not be pleat folds may be added.

Acknowledgements

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